MA2160

Test 2

Spring 2007

Name:________________________________

Instructions: You may use your calculator on the entire test. You must show enough work to justify all answers.

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1. Suppose that $I = \int_a^b f(x)\ dx$ is approximated by the midpoint rule such that MID(10)=2.00 and MID(100)=1.01.

   (a) What is the actual value of the integral?

   Solution.
   Error for MID(10)= $I - 2$.
   MID(100)= $I - \left(\frac{I-2}{10^2}\right) = 1.01$

$$I - \frac{I}{100} + \frac{2}{100} = 1.01$$

$$I\left(1 - \frac{1}{100}\right) = 1.01 - \frac{2}{100}$$

$$I = \frac{1.01 - \frac{2}{100}}{1 - \frac{1}{100}} = 1$$

□

1 [9]

   (b) Find the error in the MID(100) approximation.

   Solution.

$$1 - 1.01 = -0.01$$

□

-0.01 [5]

2. Compute the integral, if it converges, or show that it diverges. You must use limits properly for full credit.

   (a) $\int_{-1}^{2} \frac{1}{x^3}\, dx$ [10]

   Solution.

$$\int_{-1}^{2} \frac{1}{x^3}\, dx = \int_{-1}^{0} \frac{1}{x^3}\, dx + \int_{0}^{2} \frac{1}{x^3}\, dx = \lim_{a\to 0^-} \int_{-1}^{a} \frac{1}{x^3}\, dx + \lim_{b\to 0^+} \int_{b}^{0} \frac{1}{x^3}\, dx$$

$$\lim_{a\to 0^-} \left[\frac{-1}{2x^2} \Big|_{-1}^{a} \right] + \lim_{b\to 0^+} \left[\frac{-1}{2x^2} \Big|_{b}^{2} \right] = \lim_{a\to 0^-} \left[\frac{-1}{2a^2} - \frac{-1}{2(-1)^2} \right] + \lim_{b\to 0^+} \left[\frac{-1}{2(-2)^2} - \frac{-1}{2b^2} \right] = -\infty + \frac{1}{2} - \frac{1}{8} + \infty$$

   Therefore, the integral diverges. □

   (b) $\int_{-\infty}^{0} \frac{e^x}{1+e^x}\, dx$ [10]

   Solution.

$$\int_{-\infty}^{0} \frac{e^x}{1+e^x}\, dx = \lim_{a\to -\infty} \int_{a}^{0} \frac{e^x}{1+e^x}\, dx = \lim_{a\to -\infty} \left[\ln|1+e^x| \Big|_{a}^{0} \right]$$

$$= \lim_{a\to -\infty} \left[\ln|1+1| - \ln|1+e^a| \right] = \ln 2 - \ln 1 = \ln 2$$

   Therefore, the integral converges to $\ln 2$. □

3. Consider a pyramid whose base is a square with sides of length 2 meters and whose height is 5 meters.

(a) Write a Riemann sum representing the volume of the pyramid. [10]

Solution.
$\frac{w}{5-y} = \frac{2}{5}$. Then $w = \frac{2}{5}(5-y)$.

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{2}{5}(5-y_i)\right]^2 \Delta y$$

$\square$

(b) Write a definite integral representing the volume of the pyramid. You do not need to evaluate the integral. [5]

$$V = \int_0^5 \left[\frac{2}{5}(5-y)\right]^2 \, dy$$

4. Consider the solid of revolution formed by rotating the region bounded by the curve $y = x^2 + 1$, the line $x = 1$, the x-axis and the y-axis about the horizontal line $y = -2$.

   (a) Find a definite integral which represents the volume of this solid. [15]

   Solution.

$$V = \pi \int_0^1 [(x^2 + 3)^2 - 4] \, dx$$

□

   (b) Evaluate the integral to find the volume of this solid. [5]

   Solution.

$$\frac{36\pi}{5}$$

□

5. A rod of length 3 meters has density $\delta(x) = 1 + x^2$ kilograms per meter, where x is the distance (in meters) from the left end of the rod.

(a) Write a definite integral which represents the total mass of the rod and evaluate this integral. Specify the units. [5]

Solution.

$$\int_0^3 (1 + x^2)\, dx = 12 \text{ kg}$$

$\square$

(b) Find the center of mass of the rod. Specify the units. [10]

Solution.

$$\frac{\int_0^3 x(1 + x^2)\, dx}{12} = \frac{33}{16} \text{ m from the left end of the rod.}$$

$\square$

6. A water tank has the shape of a right circular cylinder with height h meters and radius R meters. Given that the tank is half full, find an expression for the work required to pump all the water over the top rim. Specify the units. [16]

Solution.

$$1000 * 9.8\pi R^2 \int_0^{h/2} (h - y)\, dy = 3675\pi R^2 h^2 \text{ joules}$$

$\square$